ŁOJASIEWICZ EXPONENTS AND RESOLUTION OF SINGULARITIES

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Abstract. We show an effective method to compute the Łojasiewicz exponent of an arbitrary sheaf of ideals of $\mathcal{O}_X$, where X is a non-singular scheme. This method is based on the algorithm of resolution of singularities.

1. Introduction

Given an analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin, the effective computation of the Łojasiewicz exponent $\mathcal{L}_0(f)$ of f is a problem that has been approached from both algebraic and analytic techniques (see for instance [1], [6], [13] or [15]). This number is defined as the infimum of those real numbers $\alpha > 0$ such that

$$\|x\|^\alpha \leqslant C\|\nabla f(x)\|,$$

for some constant $C > 0$ and all x belonging to some open neighbourhood of the origin in $\mathbb{C}^n$, where ∇f denotes the gradient of f. One of the most significant applications of $\mathcal{L}_0(f)$ is the result of B. Teissier [17, p. 280] stating that the degree of topological determinacy of f is equal to $[\mathcal{L}_0(f)] + 1$, where $[a]$ denotes the integer part of a number $a \in \mathbb{R}$. Let us denote by $j^r f$ the *r-jet* of f, that is, the sum of all terms of the Taylor expansion of f around the origin of degree $\leqslant r$. Then the degree of topological determinacy of f is defined as the minimum of those $r \geqslant 1$ such that for all $g \in \mathcal{O}_n$ verifying that $j^r f = j^r g$, we have that f and g are topologically equivalent, that is, there exists a germ of homeomorphism $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $f = g \circ \varphi$.

Let us denote by $\mathcal{O}_n$ the ring of analytic functions $f : (\mathbb{C}^n, 0) \to \mathbb{C}$. The definition of Łojasiewicz exponent of functions with an isolated singularity is extended naturally to ideals of $\mathcal{O}_n$ of finite colength. Let I be an ideal of $\mathcal{O}_n$. In this article we apply the explicit construction of a log-resolution of I given in [2] to compute effectively the Łojasiewicz exponent $\mathcal{L}_0(I)$ of I provided that I has finite colength. We consider the problem of computing $\mathcal{L}_0(I)$ in a more general setting, that is, we substitute I by a sheaf of ideals in a non-singular scheme.

As an application of the main result, we compute the Łojasiewicz exponent, and consequently the degree of topological determinacy, of a function such that $\mathcal{L}_0(f)$ can not be computed by means of the existing literature about this subject.

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2. Order functions

In this section we recall some known facts concerning the integral closure of ideals and its relation with reduced orders. We will denote by R a Noetherian ring.

Definition 2.1. Let $\bar{\mathbb{R}}_0 = \{a \in \mathbb{R} \mid a \geqslant 0\} \cup \{\infty\}$ and let us consider a function $\rho : R \to \bar{\mathbb{R}}_0$. We say that ρ is an *order function* if the following conditions hold:

(i) $\rho(f+g) \geqslant \min\{\rho(f), \rho(g)\}$, for all $f, g \in R$.
(ii) $\rho(fg) \geqslant \rho(f) + \rho(g)$, for all $f, g \in R$.
(iii) $\rho(0) = \infty$ and $\rho(1) = 0$.

Let $I \subseteq R$ be an ideal and let $f \in R$. It is well known, and also easy to prove, that the function

$$\nu_I(f) = \sup\{m \in \mathbb{N} \mid f \in I^m\}$$

is an order function. Let $J \subseteq R$ be an ideal and set

$$\nu_I(J) = \sup\{m \in \mathbb{N} \mid J \subseteq I^m\}.$$

If $f_1, \dots, f_s$ are generators of J, then it can be checked that

$$\nu_I(J) = \min\{\nu_I(f_1), \dots, \nu_I(f_s)\}.$$

Proposition 2.2. [16][14, §0.2] *Let $I \subseteq R$ be an ideal with $I \neq R$. Then the sequence*

$$\left\{ \frac{\nu_I(f^n)}{n} \right\}_{n=1}^{\infty}$$

has a limit in $\bar{\mathbb{R}}_0$. Moreover the function $\bar{\nu}_I : R \to \bar{\mathbb{R}}_0$ defined by

$$\bar{\nu}_I(f) = \lim_{n\to\infty} \frac{\nu_I(f^n)}{n}$$

is an order function.

The number $\bar{\nu}_I(f)$ is called the *reduced order of f with respect to I*. It is proved in [14] that $\bar{\nu}_I(f) \in \mathbb{Q}_+ \cup \{\infty\}$, for all $f \in R$. We will show this result using the existence of embedded desingularization of schemes and log-resolution of ideals.

Remark 2.3. The sequence $\left\{ u_n = \frac{\nu_I(f^n)}{n} \right\}_{n=1}^{\infty}$ is not an increasing sequence, in general. However, it is straightforward to see that, for any positive integer $i \geqslant 2$, the subsequence $\{u_{i^n}\}_{n=1}^{\infty}$ is increasing, so that

$$\bar{\nu}_I(f) = \lim_{n\to\infty} \frac{\nu_I(f^n)}{n} = \sup \left\{ \frac{\nu_I(f^n)}{n} \mid n \in \mathbb{N} \right\}$$

and $n\bar{\nu}_I(f) \geqslant \nu_I(f^n)$ for all n. In particular $\bar{\nu}_I(f) \geqslant \nu_I(f)$, for all $f \in R$.

Lemma 2.4. [14, 0.2.9] *Let I and J be ideals of R and let p, q be positive integers. Then*

$$\bar{\nu}_{I^p}(J^q)(x) = \frac{q}{p}\bar{\nu}_I(J).$$

For an ideal I of R we will denote by $\overline{I}$ the integral closure of I.

Lemma 2.5. [14, 1.15][11, p. 138] *Let R be a Noetherian ring and let I, J be ideals of R. If $J \subseteq \overline{I}$, then $\overline{\nu}_I(J) \geqslant 1$.*

Definition 2.6. Let $I \subseteq R$ be an ideal. We define the function $\mu_I : R \to \bar{\mathbb{R}}_0$ as

$$\mu_I(f) = \sup\left\{\frac{p}{q} \in \mathbb{Q}_+ \mid f^q \in \overline{I^p}\right\}.$$

As a consequence of [11, Proposition 10.5.2] (see also [14, §4.2]) the set of rational numbers involved in Definition 2.6 does not depend on the representatives p, q of the rational number $\frac{p}{q}$.

Let us consider the graded ring $R[T]$, with the usual graduation on T. Let $R[IT] \subseteq R[T]$ be the subring $R[IT] = \oplus_n I^n T^n$. Let $f \in R$, we have that $f \in \bar{I}$ if and only if the homogeneous element $fT \in R[T]$ is in the integral closure of the ring $R[IT]$ in $R[T]$. It is well known (see, for instance, [11, p. 95]) that this integral closure is

$$\overline{R[IT]} = \bigoplus_n \overline{I^n} T^n \subseteq R[T].$$

Lemma 2.7. *If $f^q \in \overline{I}$ and $g^q \in \overline{I}$ then $(f+g)^q \in \overline{I}$.*

Proof. By assumption f^qT and g^qT are integral over $R[IT]$. We observe that the ring extension $R[T] \subseteq R[T^{\frac{1}{q}}]$ is finite. Then $fT^{\frac{1}{q}}$ and $gT^{\frac{1}{q}}$ are integral over $R[IT] \subseteq R[T^{\frac{1}{q}}]$. Therefore $(f+g)T^{\frac{1}{q}}$ is integral over $R[IT]$. Thus $(f+g)^qT$ is integral over $R[IT]$ and we conclude that $(f+g)^q \in \overline{I}$. $\square$

Proposition 2.8. *Let I be an ideal of R. Then μ_I is an order function.*

Proof. The fact that μ_I satisfies condition (i) of Definition 2.1 follows as a direct application of Lemma 2.7. Conditions (ii) and (iii) follow easily from the definition of μ_I. $\square$

3. Resolution of singularities and integral closure.

In this section, X will denote an integral separated scheme of finite type over a field k, where the characteristic of k is zero.

If $\mathcal{I} \subseteq \mathcal{O}_X$ is a sheaf of ideals then the integral closure $\overline{\mathcal{I}}$ is a sheaf of ideals such that for every point $x \in X$, the ideal $\overline{\mathcal{I}}_x$ is the integral closure of $\mathcal{I}_x \subseteq \mathcal{O}_{X,x}$.

The next result is well known and its proof can be found, for instance, in [11, p. 133].

Lemma 3.1. *Let R be a Noetherian domain. Denote by K the field of fractions of R. Let $I \subseteq R$ be an ideal. For every valuation ring $R_v \subseteq K$ set $I_v = (IR_v) \cap R$. Then the integral closure of I is $\overline{I} = \bigcap_v I_v$, where the intersection ranges on all valuation rings in K with center in R.*

Proposition 3.2. *Let $\varphi : X' \to X$ be a proper birational morphism and let $\mathcal{I} \subseteq \mathcal{O}_X$ be a sheaf of ideals. Then $\overline{\mathcal{I}} = (\overline{\mathcal{I}\mathcal{O}_{X'}}) \cap \mathcal{O}_X$.*

Proof. It is a consequence of Lemma 3.1 and the Valuative Criterion of Properness [9, Theorem 4.7, §II]. $\square$

Definition 3.3. A *desingularization* of X is a proper birational morphism $\varphi : X' \to X$ such that

(i) X' is non-singular;
(ii) the morphism φ is an isomorphism outside the singular locus of X. That is, if $U = X \setminus Sing(X)$ and $U' = \varphi^{-1}(U)$, then $U' \cong U$ via φ.

Assume that $X \subseteq W$, where W is a non-singular scheme. An *embedded desingularization* of $X \subseteq W$ is a proper birational morphism $\Pi : W' \to W$ such that

(i) W' is non-singular;
(ii) the morphism Π is an isomorphism outside the singular locus of X. That is, if $U = W \setminus Sing(X)$ and $U' = \Pi^{-1}(U)$, then $U' \cong U$ via Π;
(iii) $W' \setminus U'$ is a simple divisor with normal crossings: $W' \setminus U' = H_1 \cup \cdots \cup H_r$;
(iv) if $X' \subseteq W'$ is the strict transform of X in W' then X' is non-singular and has only normal crossings with the divisor $W' \setminus U'$.

Definition 3.4. Let W be non-singular scheme. A *log-resolution* of an ideal $\mathcal{I} \subseteq \mathcal{O}_W$ is a proper birational morphism $\Pi : W' \to W$ such that

(i) W' is non-singular,
(ii) Π is an isomorphism outside the support of $\mathcal{I}$. If $U = W \setminus Supp(\mathcal{I})$ and $U' = \Pi^{-1}(U)$ then $U' \cong U$ via Π.
(iii) $W' \setminus U'$ is a simple divisor with normal crossings: $W' \setminus U' = H_1 \cup \cdots \cup H_r$.
(iv) The total transform of $\mathcal{I}$ in W' is a monomial with support in $W' \setminus U'$

$$\mathcal{I}\mathcal{O}_{W'} = \mathbf{I}(H_1)^{a_1} \cdots \mathbf{I}(H_r)^{a_r}. \tag{3.1}$$

Remark 3.5. It was proved by Hironaka in [10] that embedded desingularizations and log-resolutions do exist without restriction on the dimension of schemes over a field of characteristic zero. In fact, Hironaka proved that the morphism Π may be obtained as a sequence of blowing-ups along regular centers.

The proof in [10] is existential. Constructive proofs may be found in [18] and also in [3]. If the characteristic of the ground field k is positive, then resolution of singularities is an open problem for general dimension. The reader may found more details in [8]. We refer to [2] for constructive proofs of embedded desingularization of schemes, log-resolution of ideals and (non-embedded) desingularization of schemes.

Algorithms implementing resolution of singularities (in characteristic zero) in the computer are available for explicit computations. We will use the implementation of [4] available at

`http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`

and implemented in Singular [7] and Maple. There is another implementation of resolution of singularities in [5] also implemented in Singular.

Proposition 3.6. *Let us consider a log-resolution of* $\mathcal{I} \subseteq \mathcal{O}_W$*, as in Definition* 3.4*. Then*

$$\overline{\mathcal{I}^m} = \mathbf{I}(H_1)^{ma_1} \cdots \mathbf{I}(H_r)^{ma_r} \cap \mathcal{O}_W,$$

for any integer $m \geqslant 1$.

Proof. It is a consequence of Proposition 3.2 and the fact that locally principal ideals are integrally closed. □

4. The reduced order of a sheaf and Łojasiewicz exponents

As in the previous section, here X will denote an integral separated scheme of finite type over a field k.

Definition 4.1. Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two sheaves of ideals. We define two functions $\bar{\nu}_{\mathcal{I}}(\mathcal{J}) : X \to \bar{\mathbb{R}}_0$ and $\mu_{\mathcal{I}}(\mathcal{J}) : X \to \bar{\mathbb{R}}_0$ as follows

$$\bar{\nu}_{\mathcal{I}}(\mathcal{J})(x) = \bar{\nu}_{\mathcal{I}_x}(\mathcal{J}_x) = \inf_{f \in \mathcal{J}_x} \bar{\nu}_{\mathcal{I}_x}(f), \qquad \mu_{\mathcal{I}}(\mathcal{J})(x) = \mu_{\mathcal{I}_x}(\mathcal{J}_x) = \inf_{f \in \mathcal{J}_x} \mu_{\mathcal{I}_x}(f),$$

for all $x \in X$.

We say that a function $\mu : X \to \mathbb{R} \cup \{\infty\}$ is *lower-semicontinuous* if for any $\alpha \in \mathbb{R}$, the set $F_\alpha = \{x \in X \mid \mu(x) \leqslant \alpha\}$ is closed. Analogously, we say that μ is *upper-semicontinuous* when the set $G_\alpha = \{x \in X \mid \mu(x) \geqslant \alpha\}$ is closed, for all $\alpha \in \mathbb{R}$.

Lemma 4.2. *Assume that X is non-singular and that $H_1, \ldots, H_r$ are non-singular irreducible hypersurfaces having only normal crossings. Let $\lambda_1, \ldots, \lambda_r \in \bar{\mathbb{R}}_0$ and let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Let us consider the function $\lambda_i : X \to \bar{\mathbb{N}}$ given by*

$$\lambda_i(x) = \begin{cases} \lambda_i, & \text{if } x \in H_i, \\ \infty, & \text{otherwise.} \end{cases}$$

Then the function $\lambda : X \to \bar{\mathbb{N}}$ defined by $\lambda = \min\{\lambda_i \mid i = 1, \ldots, N\}$ is lower-semicontinuous.

Proof. Let $\alpha \in \bar{\mathbb{R}}_0$ and let us consider the set $F_\alpha = \{x \in X \mid \lambda(x) \leqslant \alpha\}$. We observe that F_α is the union of the hypersurfaces H_i such that $\lambda_i \leqslant \alpha$. Therefore F_α is closed and the result follows. □

Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two sheaves of ideals. Let $\Pi' : X'' \to X$ be a desingularization of X (in the sense of Definition 3.3) and let $\Pi'' : X' \to X''$ be a log-resolution of $\mathcal{I}\mathcal{O}_{X''}$ (as in Definition 3.4), so that

$$\mathcal{I}\mathcal{O}_{X'} = \mathbf{I}(H_1)^{a_1} \cdots \mathbf{I}(H_r)^{a_r}, \tag{4.1}$$

for some positive integers $a_1, \ldots, a_r$. The total transform $\mathcal{J}\mathcal{O}_{X'}$ can be expressed as

$$\mathcal{J}\mathcal{O}_{X'} = \mathbf{I}(H_1)^{b_1} \cdots \mathbf{I}(H_r)^{b_r} \mathcal{J}', \tag{4.2}$$

where $\mathcal{J}' \subseteq \mathcal{O}_{X'}$ and $\mathcal{J}' \not\subseteq \mathbf{I}(H_i)$, for all $i = 1, \ldots, r$.

Proposition 4.3. *In the setup described above, let us consider the function $\lambda = \min\{\frac{b_i}{a_i} \mid i = 1, \ldots, r\}$. Then*

$$\mu_{\mathcal{I}}(\mathcal{J})(x) = \min\big\{\lambda(x') \mid x' \in \Pi^{-1}(x)\big\},$$

for all $x \in X$, and the function $\mu_{\mathcal{I}}(\mathcal{J})$ is lower-semicontinuous.

Proof. Let p, q be positive integers. We observe that $(\mathcal{J}^q)_x \subseteq (\overline{\mathcal{I}^p})_x$ if and only if $(\mathcal{J}^q\mathcal{O}_{X'})_{x'} \subseteq (\mathcal{I}^p\mathcal{O}_{X'})_{x'}$, for all $x' \in \Pi^{-1}(x)$. Moreover, according to (4.1) and (4.2), we have the following equivalences:

$$\begin{aligned}(\mathcal{J}^q\mathcal{O}_{X'})_{x'} \subseteq (\mathcal{I}^p\mathcal{O}_{X'})_{x'} &\Longleftrightarrow (\mathbf{I}(H_1)^{qb_1}\cdots\mathbf{I}(H_r)^{qb_r}\mathcal{J}'^q)_{x'} \subseteq (\mathbf{I}(H_1)^{pa_1}\cdots\mathbf{I}(H_r)^{pa_r})_{x'}\\ &\Longleftrightarrow \left(\frac{b_i}{a_i}\right)(x') \geqslant \frac{p}{q},\ i = 1, \dots, r\\ &\Longleftrightarrow \lambda(x') \geqslant \frac{p}{q}.\end{aligned}$$

Hence

$$\mu_{\mathcal{I}}(\mathcal{J})(x) \geqslant \frac{p}{q} \Longleftrightarrow \lambda(x') \geqslant \frac{p}{q}, \quad \text{for all } x' \in \Pi^{-1}(x),$$

and we have $\mu_{\mathcal{I}}(\mathcal{J})(x) = \min\{\lambda(x') \mid x' \in \Pi^{-1}(x)\}$.

The lower-semicontinuity of $\mu_{\mathcal{I}}(\mathcal{J})$ follows from the properness of Π. □

As an immediate consequence of the previous theorem we obtain the following result.

Corollary 4.4. *The value $\mu_{\mathcal{I}}(\mathcal{J})(x)$ is rational, for every $x \in X$.*

Theorem 4.5. *Let $\mathcal{I}, \mathcal{J} \subseteq \mathcal{O}_X$ be two sheaves of ideals. Then the functions $\bar{\nu}_{\mathcal{I}}(\mathcal{J})$ and $\mu_{\mathcal{I}}(\mathcal{J})$ are equal.*

Proof. We use the same notation as in Proposition 4.3. Let us fix a point $x \in X$. First we prove that $\mu_{\mathcal{I}}(\mathcal{J}) \geqslant \bar{\nu}_{\mathcal{I}}(\mathcal{J})$.

Set $c_n = \nu_{\mathcal{I}}(\mathcal{J}^n)(x)$, for all $n \geqslant 1$. We observe that

$$\bar{\nu}_{\mathcal{I}}(\mathcal{J})(x) = \sup_{n\in\mathbb{N}} \frac{c_n}{n}.$$

By definition we have $\mathcal{J}^n \subseteq \mathcal{I}^{c_n} \subseteq \overline{\mathcal{I}^{c_n}}$, which implies that

$$\mu_{\mathcal{I}}(\mathcal{J})(x) \geqslant \frac{c_n}{n}, \quad \text{for all } n \in \mathbb{N}.$$

Therefore

$$\mu_{\mathcal{I}}(\mathcal{J})(x) \geqslant \bar{\nu}_{\mathcal{I}}(\mathcal{J}).$$

Conversely, set $\frac{p}{q} = \mu_{\mathcal{I}}(\mathcal{J})(x)$. This implies that $\mathcal{J}^q_x \subseteq \overline{\mathcal{I}^p}_x$. By Lemma 2.5 we have that $\bar{\nu}_{\mathcal{I}^p}(\mathcal{J}^q)(x) \geqslant 1$ and from Lemma 2.4 we obtain $\bar{\nu}_{\mathcal{I}}(\mathcal{J})(x) \geqslant \frac{p}{q}$. □

Corollary 4.6. *The value $\bar{\nu}_{\mathcal{I}}(\mathcal{J})(x)$ is rational, for every $x \in X$.*

Definition 4.7. Let X be a scheme as above with structure of complex variety. Let $\mathcal{I} \subseteq \mathcal{O}_X$ be a coherent sheaf of ideals and $K \subseteq X$ be a compact set. Let $f \in \Gamma(X, \mathcal{O}_X)$. The *Łojasiewicz exponent of f with respect to $\mathcal{I}$ at K*, denoted by $\theta_K(f, \mathcal{I})$, is defined as the infimum of those $\theta \in \mathbb{R}_+$ such that there exists an open set $U \subseteq \mathbb{C}^n$ such that $K \subseteq U$ and a constant $C \geqslant 0$ such that

$$|f(x)|^\theta \leqslant C \cdot \sup_{g\in\Gamma(U,\mathcal{I})} |g(x)|,$$

for all $x \in U$.

If $\mathcal{J} \subseteq \mathcal{O}_X$ is a sheaf of ideals, then

$$\theta_K(\mathcal{J},\mathcal{I}) = \sup_{f\in\Gamma(X,\mathcal{J})} \theta_K(f,\mathcal{I}).$$

Theorem 4.8. [14, 6.3] *Under the hypothesis of the previous definition we have*

$$\theta_K(\mathcal{J},\mathcal{I}) = \frac{1}{\bar{\nu}_{\mathcal{I}}(\mathcal{J})(K)},$$

where $\bar{\nu}_{\mathcal{I}}(\mathcal{J})(K) = \min\{\bar{\nu}_{\mathcal{I}}(\mathcal{J})(x) \mid x \in K\}$.

As a direct consequence of the previous theorem and of Corollary 4.6 we obtain that the Łojasiewicz exponent $\theta_K(\mathcal{J},\mathcal{I})$ is a rational number.

Definition 4.9. Let $\mathcal{I},\mathcal{J} \subseteq \mathcal{O}_X$ be two sheaves of ideals. We define the function $\theta(\mathcal{J},\mathcal{I}) : X \to \mathbb{Q}$ as follows:

$$\theta(\mathcal{J},\mathcal{I})(x) = \theta_{\{x\}}(\mathcal{J},\mathcal{I}),$$

for all $x \in X$.

From Proposition 4.3 and Theorem 4.8 we obtain that the function $\theta(\mathcal{J},\mathcal{I}) : X \to \mathbb{Q}$ is upper-semicontinuous.

5. Computation of Łojasiewicz exponents for isolated singularities.

Let W be an scheme with structure a regular analytic variety. Let $\mathcal{I}$ be a sheaf of ideals in $\mathcal{O}_W$ such that $Supp(\mathcal{I}) = \{x\}$, where $x \in W$. We define the *Łojasiewicz exponent of* $\mathcal{I}$ *at* x as $\mathcal{L}_x(\mathcal{I}) = \theta(\mathcal{J},\mathcal{I})(x)$ where $\mathcal{J}$ is the sheaf of ideals

$$\mathcal{J}_y = \begin{cases} \mathfrak{m}_x & \text{if} \quad y = x \\ 1 & \text{if} \quad y \neq x. \end{cases}$$

Theorem 5.1. *The Łojasiewicz exponent of* $\mathcal{I}$ *is determined by the total transform of* $\mathfrak{m}_x$ *via the log-resolution of* $\mathcal{I}$.

Proof. Let us consider a log-resolution of $\mathcal{I}$ as in Definition 3.4. The morphism $W' \to W$ is a sequence of blowing-ups along regular centers:

$$W = W_0 \longleftarrow W_1 \longleftarrow \cdots \longleftarrow W_r = W'.$$

We observe that the first blowing-up must have $Supp(\mathcal{I})$ as center. Therefore $\mathcal{J}\mathcal{O}_{W_1} = \mathfrak{m}_x\mathcal{O}_{W_1} = I(H_1)$ and the total transform of $\mathfrak{m}_x$ is a monomial, that is

$$\mathfrak{m}_x\mathcal{O}_{W'} = \mathbf{I}(H_1)^{b_1}\cdots\mathbf{I}(H_r)^{b_r},$$

for some positive integers $b_1,\ldots,b_r$.

Let us suppose that the total transform of $\mathcal{I}$ in W' is written as in (3.1). Then, we obtain the following equivalences:

$$\begin{aligned} \mathfrak{m}_x^p \subseteq \overline{\mathcal{I}^q} \quad &\Longleftrightarrow \quad \mathbf{I}(H_1)^{pb_1}\cdots\mathbf{I}(H_r)^{pb_r} \subseteq \mathbf{I}(H_1)^{qa_1}\cdots\mathbf{I}(H_r)^{qa_r} \\ &\Longleftrightarrow \quad pb_i \geqslant qa_i, \quad i = 1,\ldots,r \\ &\Longleftrightarrow \quad \frac{p}{q} \geqslant \frac{a_i}{b_i}, \quad i = 1,\ldots,r. \end{aligned}$$

Then, we conclude that

$$\mathcal{L}_x(\mathcal{I}) = \max\left\{\frac{a_i}{b_i},\ i = 1, \dots, r\right\}. \tag{5.1}$$

□

By (5.1), the problem of computing $\mathcal{L}_x(\mathcal{I})$ reduces to determine the integers a_i, b_i, for $i = 1, \dots, r$, which in turn, come from determining the total transform of $\mathfrak{m}_x$ via the log-resolution of $\mathcal{I}$. Next we expose some examples in the ring $\mathcal{O}_n$ of holomorphic gems $f : (\mathbb{C}^n, 0) \to \mathbb{C}$.

Example 1. Let us consider the ideal I of $\mathcal{O}_3$ generated by the polynomials

$$g_1 = x^4 + xyz + y^4$$
$$g_2 = xy^2z$$
$$g_3 = y^5 + z^5.$$

Then, applying relation (5.1), it follows that $\mathcal{L}_0(I) = 5 + \frac{5}{6}$. Let us denote by $e(I)$ the Samuel multiplicity of I. The same value for $\mathcal{L}_0(I)$ is obtained by following the approach explained in Section 4 of [1], since $e(I)$ equals the Rees mixed multiplicity of the ideals $I_1 = \langle x^4, xyz, y^4\rangle$, $I_2 = \langle xy^2z\rangle$ and $I_3 = \langle y^5, z^5\rangle$, which is equal to 80.

Example 2. Let us consider the function $f \in \mathcal{O}_3$ given by $f(x, y, z) = y^6 + z^4 + x(x-3z)^2$ and let us denote by $\mu(f)$ the Milnor number of f. We observe that f is a Newton degenerate function in the sense of [12]. Moreover $\mu(f) = 25$, whereas the Newton number of the Newton polyhedron of f is equal to 20. Therefore, the Łojasiewicz exponent of f can not be computed using the technique explained in [1] via mixed multiplicities of monomial ideals.

Using relation (5.1) we obtain

$$\mathcal{L}_0(\nabla f) = 5.$$

Therefore, by virtue of [17], the degree of topological determinacy of f is given by

$$[\mathcal{L}_0(\nabla f)] + 1 = 6.$$

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